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SOME USEFUL PROPERTIES OF AMBIGUITY FUNCTION ASSOCIATED WITH LINEAR CANONICAL TRANSFORM

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Abstract

The ambiguity function associated with the linear canonical transform (LCT) is a generalization of the one-dimensional ambiguity function using the linear canonical transform, called the linear canonical ambiguity function (LCAF). We first investigate its basic properties such as the complex conjugation, translation and modulation. These properties are extensions of the corresponding versions of the classical

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ambiguity function. Using the basic relationship between the LCT and LCAF, we derive the inversion and Moyal formulas for the LCAF. Based on a convolution theorem for the LCT, we propose the convolution theorem related to LCAF. Finally, through simulation example, we demonstrate how the proposed convolution generalizes the formulation of the classical ambiguity function convolution.

1. Introduction

The linear canonical transform (LCT) [4, 6, 7] is a linear integral transform with three free parameters which has found many applications in several areas, including signal processing and optics. It can be regarded as generalization of many transforms such as the Fourier transform, Laplace transform, the fractional Fourier transform, the Fresnel transform and the other transforms. Many properties of this transform are already known, including shift, modulation, and uncertainty principle [9, 11, 14, 16-18].

Recently, many efforts have been devoted to extend various types of transform to the linear canonical transform. Tao et al. [13] studied the short-time fractional Fourier transform which are generalization of the short-time Fourier transform to the LCT. Some applications of the extended transform such as the estimations of the time-of-arrival (TOA), pulse width (PW) of chirp signals, and the STFRFD filtering are also discussed. Fan et al. [3] proposed an extension of the quaternion Wigner-Ville distribution to the LCT. Some useful properties of the generalized transform were also studied. In [6], the authors proposed the Wigner-Ville distribution associated LCT and established its convolution and correlation theorems. In [19], the authors proposed the generalization of the classical wavelet transform to the LCT domain and its application to blind image watermarking. The authors [20] discussed the convolution and correlation theorems for the 2-D LCT, which are the generalization of the convolution and correlation theorem for 2-D Fourier transform. In [15], the generalization of the classical ambiguity function to the LCT was recently presented. Some general properties of the generalized transform were shortly studied.

Therefore, the purpose of this present paper is to introduce the linear canonical ambiguity function (LCAF). The transform is obtained by replacing the Fourier kernel with the LCT kernel in the AF definition. We investigate several basic properties of the LCAF such as complex conjugation, shift and modulation. These properties are very important for their applications in digital signal and image processing. Using the relationship between the LCT and LCAF we derive the inversion and Moyal formulas related to the LCAF. Based on convolution theorem for the LCT we define the convolution of the LCAF and find its convolution theorem. Finally, we present an application of the LCAF convolution theorem to demonstrate how the proposed convolution generalizes the formulation of the ambiguity function convolution theorem.

2. Preliminaries

2.1. Linear canonical transform

The LCT is firstly proposed by Moshinsky and Quesnee [4] and Collins [8]. Here we briefly introduce the LCT definition.

Definition 2.1 (LCT)**.** Let $A = (a, b, c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter such that $\det(A) = ad - bc = 1.$ The LCT of a signal $f \in L^2(\mathbb{R})$ is defined by

$$L_A(f)(\omega) = \begin{cases} \int_{-\infty}^{\infty} f(x) K_A(\omega, x) dx, & b \neq 0, \\ \sqrt{d}\, e^{i\left(\frac{cd}{2}\right)\omega^2} f(d\omega), & b = 0, \end{cases} \tag{1}$$

where $K_A(\omega, x)$ is so-called kernel function of the LCT given by

$$K_A(\omega, x) = \frac{1}{\sqrt{2\pi bi}} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)}. \tag{2}$$

It also should be remembered, when $b = 0,$ the LCT of a signal is essentially a chirp multiplication. Therefore, in this work, we always assume $b \neq 0.$ The inverse transform of the LCT is given by

$$f(x) = \int_{-\infty}^{\infty} L_A(f)(\omega) K_{A^{-1}}(\omega, x) d\omega$$

$$= \frac{1}{\sqrt{-2\pi bi}} \int_{-\infty}^{\infty} L_A(f)(\omega) e^{i\frac{1}{2}\left[\left(\frac{-d}{b}\right)\omega^2 + \left(\frac{2}{b}\right)x\omega - \left(\frac{a}{b}\right)x^2\right]} d\omega, \quad (3)$$

where the inverse of matrix parameter A is denoted by A^{-1} and $A^{-1} = (d, -b, -c, a)$.

An important property of the LCT is the Parseval's formula which will be used to establish Moyal's formula for the linear canonical ambiguity function (LCAF)

$$(f, g) = (L_A(f), L_A(g)), \quad (4)$$

for all $f, g \in L^2(\mathbb{R})$. In particular, for $f = g$ we obtain the Plancherel's formula for the LCT as

$$\| f \|_2 = \| L_A(f) \|_2,$$

where L^p norm is defined by

$$\| f \|_p = \left(\int_{-\infty}^{\infty} | f(x) |^p dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

2.2. Ambiguity function

In this subsection, we briefly discuss the important properties of the ambiguity function. For more details, we refer the reader to [1, 2, 10, 11].

Definition 2.2 (Ambiguity function). If two functions $f, g \in L^2(\mathbb{R})$, the cross ambiguity function of f and g is defined by

$$A_{f,g}(t, \omega) = \int_{-\infty}^{\infty} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-i\omega x} dx. \quad (5)$$

Some basic properties of the classical AF are summarized as follows. Let $f, g \in L^2(\mathbb{R})$. Denote by τ_a is the shift operator defined by $\tau_k f(x) = f(x-k)$ and by $\mathbb{M}_{\omega_0}$ is a modulation operator defined $\mathbb{M}_{\omega_0} f(x) = e^{i\omega_0 x} f(x)$.

1. Complex conjugation

$$\overline{A_{f,g}(t, \omega)} = W_{g,f}(-t, -\omega).$$

2. Translation

$$A_{\tau_k f, \tau_k g}(t, \omega) = e^{-i\omega_0 k} A_{f,g}(t, \omega).$$

3. Modulation

$$A_{\mathbb{M}_{\omega_0} f, \mathbb{M}_{\omega_0} g}(t, \omega) = e^{-i\omega_0 k} A_{f,g}(t, \omega - \omega_0).$$

4. Moyal's formula

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_{f_1, g_1}(t, \omega) \overline{A_{f_2, g_2}(t, \omega)} dt d\omega = (f_1, f_2) \overline{(g_1, g_2)}.$$

5. Inversion formula

$$f(t) = \frac{1}{2\pi \overline{g(0)}} \int_{-\infty}^{\infty} A_{f,g}\left(\frac{t}{2}, \omega\right) e^{i\omega t} d\omega,$$

provided $\overline{g(0)} \neq 0$.

3. Linear Canonical Ambiguity Function (LCAF)

3.1. Definition of LCAF

Based on the definition of the classical ambiguity function associated with the Fourier transform, we obtain a definition of the linear canonical ambiguity function (LCAF) by replacing the kernel of the FT with the kernel of the LCT in the classical AF definition. This definition is similar to the one proposed in [15]. Therefore, in this paper, we shall investigate more properties of the LCAF.

Definition 3.1. If $f, g \in L^2(\mathbb{R})$, then the cross linear canonical ambiguity function (LCAF) of two functions *f* and *g* is defined as

$$\mathcal{AF}^A_{f,g}(t, \omega) = \int_{-\infty}^{\infty} f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)} \frac{1}{\sqrt{2\pi b i}} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)} dx. \quad (6)$$

Suppose the kernel of the LCT with parameter A is $K_A(\omega, x)$ defined in (2). Then (5) takes the form

$$\mathcal{AF}^A_{f,g}(t, \omega) = \int_{-\infty}^{\infty} h_{f,g}(x, t) K_A(\omega, x) dx, \quad (7)$$

where

$$h_{f,g}(x, t) = f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)}. \quad (8)$$

If $f = g$, then $\mathcal{AF}^A_{f,f}(t, \omega) = \mathcal{AF}^A_f(t, \omega)$ is called the *auto linear canonical ambiguity function*. Often both the cross LCAF and the auto LCAF are usually referred to simply as the LCAF.

It follows from Definition 3.1 that the cross LCAF is the LCT of the function $h_{f,g}(x, t)$ with respect to *x*. In other words,

$$\mathcal{AF}^A_{f,g}(t, \omega) = L_A\{h_{f,g}(x, t)\}(\omega).$$

The following result describes an inequality related to the LCAF.

Theorem 3.2. *Suppose that* $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$ *with* $1/p + 1/q = 1$. *Then we have*

$$|\mathcal{AF}^A_{f,g}(t, \omega)| \leq \frac{1}{\sqrt{2\pi b}} \| f \|_p \| g \|_q. \quad (9)$$

Proof. A straightforward computation shows that

$$
\begin{aligned}
&|\mathcal{AF}_{f,g}^{A}(t,\omega)| \\
&= \left| \int_{-\infty}^{\infty} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}\left(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2\right)}dx \right| \\
&\le \left(\int_{-\infty}^{\infty}\left| f\left(x+\frac{t}{2}\right)\right|^{p}dx\right)^{1/p} \\
&\quad\times\left(\int_{-\infty}^{\infty}\left|\overline{g\left(x-\frac{t}{2}\right)}\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}\left(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2\right)}\right|^{q}dx\right)^{1/q} \\
&= \frac{1}{\sqrt{2\pi b}}\left(\int_{-\infty}^{\infty}\left| f\left(x+\frac{t}{2}\right)\right|^{p}dx\right)^{1/p}\left(\int_{-\infty}^{\infty}\left|\overline{g\left(x-\frac{t}{2}\right)}\right|^{q}dx\right)^{1/q} \\
&= \frac{1}{\sqrt{2\pi b}}\left(\int_{-\infty}^{\infty}| f(y)|^{p}dy\right)^{1/p}\left(\int_{-\infty}^{\infty}|\overline{g(y)}|^{q}dy\right)^{1/q}.
\end{aligned}
$$

Hence, the result follows.

Observe first that for $p = q = 2$, equation (9) will reduce to

$$|\mathcal{AF}_{f,g}^{A}(t,\omega)| \le \frac{1}{\sqrt{2\pi b}}\| f \|_2 \| g \|_2,$$

which shows that $\mathcal{AF}_{f,g}^{A}(t,\omega)$ is bounded on $L^2(\mathbb{R})$.

Example 1. Find the LCAF of a Gaussian signal

$$f(t) = (\pi\sigma^2)^{-1/4}e^{-t^2/2\sigma^2}.$$

Applying the definition of the LCAF (5) gives

$$\mathcal{AF}_f^A(t, \omega) = (\pi\sigma^2)^{-1/2} \frac{1}{\sqrt{2\pi bi}}$$

$$\times \int_{-\infty}^{\infty} e^{-\left(x+\frac{t}{2}\right)^2 \Big/ 2\sigma^2} e^{-\left(x-\frac{t}{2}\right)^2 \Big/ 2\sigma^2} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)} dx$$

$$= (\pi\sigma^2)^{-1/2} \frac{1}{\sqrt{2\pi bi}} e^{-t^2/4\sigma^2} \int_{-\infty}^{\infty} e^{-x^2/\sigma^2} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)} dx.$$

Applying the LCT of the Gaussian function, which can be found in [5], we finally arrive at

$$\mathcal{AF}_f^A(t, \omega) = (\pi\sigma^2)^{-1/2} \frac{1}{\sqrt{2\pi bi}} e^{-t^2/4\sigma^2} \frac{\sqrt{2}\sigma}{\sqrt{2\sigma^2 a + bi}} e^{\frac{\omega^2}{2}\left(\frac{2c\sigma^2 + id}{b - 2ia\sigma^2}\right)}.$$

3.2. Useful properties of LCAF

In this subsection, we discus some useful properties of the LCAF and theirs proofs. We see that the most of them are extensions of the corresponding version of the classical ambiguity function (AF) with the some modifications.

Theorem 3.3 (*Complex conjugation*)**.** *For any function* $f, g \in L^2(\mathbb{R})$, *we have*

$$\overline{\mathcal{AF}_{f,g}^A(t, \omega)} = \mathcal{AF}_{g,f}^{A^{-1}}(-t, \omega). \tag{10}$$

Proof. Applying the definition of the LCAF (6) and inverse of the matrix parameter $A = (a, b, c, d)$, we easily obtain

$$\overline{\mathcal{AF}_{f,g}^A(t, \omega)} = \int_{-\infty}^{\infty} \overline{f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)} \frac{1}{\sqrt{2\pi bi}} e^{l\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)}} dx$$

$$= \frac{1}{\sqrt{-2\pi bi}} \int_{-\infty}^{\infty} g\left(x - \frac{t}{2}\right) \overline{f\left(x + \frac{t}{2}\right)} e^{-i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)} dx$$

$$= \int_{-\infty}^{\infty} g\left(x - \frac{t}{2}\right)\overline{f\left(x + \frac{t}{2}\right)} K_{A^{-1}}(\omega, x)dx$$

$$= \mathcal{AF}_{g, f}^{A^{-1}}(-t, \omega).$$

Theorem 3.4 (Translation)**.** *Suppose that* $f, g \in L^2(\mathbb{R})$. *Then we have*

$$\mathcal{AF}_{\tau_k f, \tau_k g}^{A}(t, \omega) = e^{ick\omega} e^{-i\frac{ack^2}{2}} \mathcal{AF}_{f, g}^{A}(t, \omega - ka). \tag{11}$$

Proof. It follows from the definition of the LCAF (6) that

$$\mathcal{AF}_{\tau_k f, \tau_k g}^{A}(t, \omega) = \int_{-\infty}^{\infty} f\left(x - k + \frac{t}{2}\right)\overline{g\left(x - k - \frac{t}{2}\right)} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)} dx.$$

Letting $x - k = y$ and then applying shift property of the LCT, we immediately obtain

$$\mathcal{AF}_{\tau_k f, \tau_k g}^{A}(t, \omega)$$

$$= \int_{-\infty}^{\infty} f\left(y + \frac{t}{2}\right)\overline{g\left(y - \frac{t}{2}\right)} e^{i\frac{1}{2}\left(\frac{a}{b}(y+k)^2 - \frac{2}{b}(y+k)\omega + \frac{d}{b}\omega^2\right)} dy$$

$$= e^{ick\omega} e^{-i\frac{ack^2}{2}} \mathcal{AF}_{f, g}^{A}(t, \omega - ka),$$

which was to be proved.

Theorem 3.5 (Modulation)**.** *For any function* $f, g \in L^2(\mathbb{R})$, *we have*

$$\mathcal{AF}_{\mathbb{M}_{\omega_0} f, \mathbb{M}_{\omega_0} g}^{A}(t, \omega) = e^{i\omega_0 t} \mathcal{AF}_{f, g}^{A}(t, \omega). \tag{12}$$

Proof. Simple calculations show that

$$\mathcal{AF}_{\mathbb{M}_{\omega_0} f, \mathbb{M}_{\omega_0} g}^{A}(t, \omega)$$

$$= \int_{-\infty}^{\infty} e^{i\omega_0\left(x + \frac{t}{2}\right)} f\left(x + \frac{t}{2}\right) \overline{e^{-i\omega_0\left(x - \frac{t}{2}\right)} g\left(x - \frac{t}{2}\right)} K_A(\omega, x)dx$$

$$= \frac{1}{\sqrt{2\pi bi}} e^{i\omega_0 t} \int_{-\infty}^{\infty} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)} dx$$

$$= e^{i\omega_0 t} \mathcal{AF}_{f,g}^{A}(t, \omega).$$

Theorem 3.6 (Modulation and translation)**.** *Let* $f, g \in L^2(\mathbb{R})$ *be two complex functions*. *Then we get*

$$\mathcal{AF}_{\mathbb{M}_{\omega_0}\tau_k f, \mathbb{M}_{\omega_0}\tau_k g}^{A}(t, \omega) = e^{i\omega_0 t} \mathcal{AF}_{f,g}^{A}(t, \omega - ka), \tag{13}$$

$$\mathcal{AF}_{\mathbb{M}_{\omega_0}\tau_k f, \tau_k g}^{A}(t, \omega) = e^{i\frac{1}{2}\omega_0 t} e^{i\frac{1}{2}\omega_0 e^{-i(ack^2 + bd\omega_0^2)/2 + i(ck + d\omega_0)\omega - ibck\omega_0}}$$

$$\times \mathcal{AF}_{f,g}^{A}(t, \omega - ak - \omega_0 b), \tag{14}$$

$$\mathcal{AF}_{\tau_k f, \mathbb{M}_{\omega_0}\tau_k g}^{A}(t, \omega) = e^{i\frac{1}{2}\omega_0 t} e^{i\frac{1}{2}\omega_0 e^{-i(ack^2 + bd\omega_0^2)/2 + i(ck + d\omega_0)\omega - ibck\omega_0}}$$

$$\times \mathcal{AF}_{f,g}^{A}(t, \omega - ak + \omega_0 b). \tag{15}$$

Proof. For (13), an easy computation yields

$$\mathcal{AF}_{\mathbb{M}_{\omega_0}\tau_k f, \mathbb{M}_{\omega_0}\tau_k g}^{A}(t, \omega)$$

$$= \int_{-\infty}^{\infty} e^{i\omega_0\left(x + \frac{t}{2}\right)} f\left(x + \frac{t}{2} - k\right) e^{-i\omega_0\left(x - \frac{t}{2}\right)} \overline{g\left(x - \frac{t}{2} - k\right)} K_A(\omega, x) dx$$

$$= \frac{1}{\sqrt{2\pi bi}} e^{i\omega_0 t} \int_{-\infty}^{\infty} f\left(x + \frac{t}{2} - k\right) \overline{g\left(x - \frac{t}{2} - k\right)} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}\omega x + \frac{d}{b}\omega^2\right)} dx$$

$$= e^{i\omega_0 t} e^{ick\omega} e^{-i\frac{ack^2}{2}} \mathcal{AF}_{f,g}^{A}(t, \omega - ka),$$

where the last line of above expression follows from translation property of the LCT. For (14), we have, by definition

$$\mathcal{AF}^{A}_{\mathbb{M}_{\omega_0}\tau_k f,\, \tau_k g}(t,\, \omega)$$

$$= \int_{-\infty}^{\infty} e^{i\omega_0\left(x+\frac{t}{2}\right)} f\left(x - k + \frac{t}{2}\right)\overline{g\left(x - k - \frac{t}{2}\right)} K_A(\omega,\, x)dx$$

$$= \frac{1}{\sqrt{2\pi bi}}\int_{-\infty}^{\infty} e^{i\omega_0 x} e^{i\frac{1}{2}\omega_0 t} f\left(x - k + \frac{t}{2}\right)\overline{g\left(x - k - \frac{t}{2}\right)} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}\omega x + \frac{d}{b}\omega^2\right)} dx$$

$$= \frac{1}{\sqrt{2\pi bi}} e^{i\frac{1}{2}\omega_0 t}\int_{-\infty}^{\infty} f\left(x - k + \frac{t}{2}\right)\overline{g\left(x - k - \frac{t}{2}\right)} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}(\omega - \omega_0 b)x + \frac{d}{b}\omega^2\right)} dx$$

$$= e^{i\frac{1}{2}\omega_0 t} e^{-i(ack^2 + bd\omega_0^2)/2 + i(ck + d\omega_0)\omega - ibck\omega_0}\mathcal{AF}^{A}_{f,\, g}(t,\, \omega - ak - b\omega_0).$$

For (15), an application of modulation and translation properties of the LCT we easily obtain

$$\mathcal{AF}^{A}_{\tau_k f,\, \mathbb{M}_{\omega_0}\tau_k g}(t,\, \omega)$$

$$= \int_{-\infty}^{\infty} f\left(x + \frac{t}{2} - k\right) e^{-i\omega_0\left(x-\frac{t}{2}\right)}\overline{g\left(x - \frac{t}{2} - k\right)} K_A(\omega,\, x)dx$$

$$= e^{i\frac{1}{2}\omega_0 t} e^{-i(ack^2 + bd\omega_0^2)/2 + i(ck + d\omega_0)\omega - ibck\omega_0}\mathcal{AF}^{A}_{f,\, g}(t,\, \omega - ak + b\omega_0).$$

Theorem 3.7 (Reconstruction formula)**.** *For any function* $f,\, g \in L^2(\mathbb{R})$, *we have*

$$f(t) = \frac{1}{\overline{g(0)}}\int_{-\infty}^{\infty} \mathcal{AF}^{A}_{f,\, g}\left(\frac{t}{2},\, \omega\right) K_{A^{-1}}(t,\, \omega)d\omega. \tag{16}$$

Proof. It directly follows from (7) that

$$\mathcal{AF}^{A}_{f,\, g}(t,\, \omega) = L_A\{h_{f,\, g}(x,\, t)\}(\omega) = \int_{-\infty}^{\infty} h_{f,\, g}(x,\, t) K_A(\omega,\, x)dx.$$

Applying the inverse of the LCT (3), we get

$$h_{f,g}(x, t) = \int_{-\infty}^{\infty} \mathcal{AF}_{f,g}^{A}(t, \omega) K_{A^{-1}}(x, \omega) d\omega,$$

and thus

$$f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)} = \int_{-\infty}^{\infty} \mathcal{AF}_{f,g}^{A}(t, \omega) K_{A^{-1}}(x, \omega) d\omega.$$

Taking the specific value, $x = \frac{t}{2}$ and the above yields

$$f(t)\overline{g(0)} = \int_{-\infty}^{\infty} \mathcal{AF}_{f,g}^{A}(t, \omega) K_{A^{-1}}\left(\frac{t}{2}, \omega\right) d\omega,$$

or equivalently,

$$f(t) = \frac{1}{\overline{g(0)}} \int_{-\infty}^{\infty} \mathcal{AF}_{f,g}^{A}(t, \omega) K_{A^{-1}}\left(\frac{t}{2}, \omega\right) d\omega.$$

Theorem 3.8 (Moyal's formula)**.** *For complex functions* $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$, *then the following result holds*:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{AF}_{f_1,g_1}^{A}(t, \omega)\overline{\mathcal{AF}_{f_2,g_2}^{A}(t, \omega)} d\omega dt = 2(f_1, f_2)\overline{(g_1, g_2)}. \qquad (17)$$

In particular, *we have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} | \mathcal{AF}_{f,g}^{A}(t, \omega) |^2 d\omega dt = 2\| f \|_2 \| g \|_2 \qquad (18)$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{AF}_{f}^{A}(t, \omega)\overline{\mathcal{AF}_{g}^{A}(t, \omega)} d\omega dt = 2|(f, g)|^2. \qquad (19)$$

Proof. Applying Parseval's formula of the LCT (4) to ω-integral into the left-hand side of (17) yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{AF}_{f_1, g_1}^{A}(t, \omega)\overline{\mathcal{AF}_{f_2, g_2}^{A}(t, \omega)}d\omega dt$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} L_A\{h_{f_1, g_1}(x, t)\}(\omega)\overline{L_A\{h_{f_2, g_2}(x, t)\}(\omega)}d\omega\right)dt$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h_{f_1, g_1}(x, t)\overline{h_{f_2, g_2}(x, t)}dx\right)dt.$$

Therefore, we further get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{AF}_{f_1, g_1}^{A}(t, \omega)\overline{\mathcal{AF}_{f_2, g_2}^{A}(t, \omega)}d\omega dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(x + \frac{t}{2}\right)\overline{g_1\left(x - \frac{t}{2}\right)}g_2\left(x - \frac{t}{2}\right)\overline{f_2\left(x + \frac{t}{2}\right)}dtdx.$$

Making the change of variables $y = x + \frac{t}{2}$ and $z = x - \frac{t}{2}$ and applying Fubini's theorem, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{AF}_{f_1, g_1}^{A}(t, \omega)\overline{\mathcal{AF}_{f_2, g_2}^{A}(t, \omega)}d\omega dt$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(y)\overline{g_1(z)}g_2(z)\overline{f_2(y)}dydz$$

$$= 2\int_{-\infty}^{\infty} f_1(y)\overline{f_2(y)}dy\int_{-\infty}^{\infty}\overline{g_1(z)}g_2(z)dz$$

$$= 2(f_1, f_2)\overline{(g_1, g_2)}.$$

The convolution is a fundamental signal processing algorithm in the theory of linear time-invariant (LTI) systems. In engineering, it has been widely used for various template matching. In the following, we first define the convolution for the LCT (see, for example, [16, 17]. It is the extension of the convolution definitions from the FT domain to the LCT domain. We present the convolution definition associated with LCAF and then establish convolution theorem related to the LCAF.

Definition 3.9 (LCT convolution)**.** For two complex functions $f, g \in L^2(\mathbb{R})$, we define the convolution operator of the LCT as

$$(f \diamond g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)W(x, t)dx, \tag{20}$$

where the weight function $W(x, t) = e^{i\frac{a}{b}2x(x-t)}$.

As an easy consequence of the above definition, we get the following important result.

Theorem 3.10 (LCAF convolution)**.** *Let* $f, g \in L^2(\mathbb{R})$ *be complex signals*. *Then we have*

$$\begin{aligned}&\mathcal{AF}^A_{f \diamond g}(t, \omega)\\ &= \int_{-\infty}^{\infty} \mathcal{AF}_f(u, \omega)\mathcal{AF}_g(t-u, \omega)e^{-i\frac{d}{b}\omega^2}e^{-i\frac{a}{b}(4u(t-u))}du.\end{aligned} \tag{21}$$

When $A = (a, b, c, d) = (0, 1, -1, 0)$, *equation* (21) *will reduce to*

$$\mathcal{AF}^A_{f \diamond g}(t, \omega) = \int_{-\infty}^{\infty} \mathcal{AF}_f(u, \omega)\mathcal{AF}_g(t-u, \omega)du. \tag{22}$$

Proof. By the LCAF definition (6), we easily obtain

$$\begin{aligned}&\mathcal{AF}^A_{f \diamond g}(t, \omega)\\ &= \int_{-\infty}^{\infty} (f \diamond g)\left(\tau + \frac{t}{2}\right)\overline{(f \diamond g)\left(\tau - \frac{t}{2}\right)}e^{i\frac{1}{2}\left(\frac{a}{b}\tau^2 - \frac{2}{b}\tau\omega + \frac{d}{b}\omega^2\right)}d\tau\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g\left(\tau + \frac{t}{2} - x\right)e^{i\frac{a}{b}2x\left(x - \left(\tau + \frac{t}{2}\right)\right)}dx\\ &\quad\times \int_{-\infty}^{\infty} \overline{f(y)g\left(\tau - \frac{t}{2} - y\right)e^{i\frac{a}{b}2y\left(y - \left(\tau - \frac{t}{2}\right)\right)}}dy\\ &\quad\times e^{i\frac{1}{2}\left(\frac{a}{b}\tau^2 - \frac{2}{b}\tau\omega + \frac{d}{b}\omega^2\right)}d\tau.\end{aligned} \tag{23}$$

Putting $x = u + \frac{p}{2}$, $y = u - \frac{p}{2}$ and $\tau = p + q$, we immediately get

$$\mathcal{AF}^{A}_{f \Diamond g}(t, \omega)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(u + \frac{p}{2}\right) g\left(t + \frac{\tau}{2} - \left(u + \frac{p}{2}\right)\right) e^{-i\frac{a}{b}2\left(u+\frac{p}{2}\right)\left(t+\frac{\tau}{2}-\left(u+\frac{p}{2}\right)\right)}$$

$$\times \overline{f\left(u - \frac{p}{2}\right)}\,\overline{g\left(t - \frac{\tau}{2} - \left(u - \frac{p}{2}\right)\right)}$$

$$\times e^{-i\frac{a}{b}2\left(u-\frac{p}{2}\right)\left(t-\frac{\tau}{2}-\left(u-\frac{p}{2}\right)\right)} e^{i\frac{1}{2}\left(\frac{a}{b}\tau^2 - \frac{2}{b}\tau\omega + \frac{d}{b}\omega^2\right)} dpdqdu$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(u + \frac{p}{2}\right) g\left(t - u + \frac{q}{2}\right)$$

$$\times \overline{f\left(u - \frac{p}{2}\right)}\,\overline{g\left(t - u - \frac{q}{2}\right)} e^{-i\frac{a}{b}2\left(u+\frac{p}{2}\right)\left(t-u+\frac{q}{2}\right)}$$

$$\times e^{-i\frac{a}{b}2\left(u-\frac{p}{2}\right)\left(t-u-\frac{q}{2}\right)} e^{i\frac{1}{2}\left(\frac{a}{b}(p+q)^2 - \frac{2}{b}(p+q)\omega + \frac{d}{b}\omega^2\right)} dpdqdu$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(u + \frac{p}{2}\right) g\left(t - u + \frac{q}{2}\right)$$

$$\times \overline{f\left(u - \frac{p}{2}\right)}\,\overline{g\left(t - u - \frac{q}{2}\right)} e^{-i\frac{a}{b}(4u(t-u)+pq)}$$

$$\times e^{i\frac{1a}{2b}p^2} e^{i\frac{1}{2}2pq} e^{i\frac{1a}{2b}q^2} e^{-i\frac{p}{b}\omega} e^{-i\frac{q}{b}\omega} e^{i\frac{1d}{2b}\omega^2} dpdqdu$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(u + \frac{p}{2}\right) \overline{f\left(u - \frac{p}{2}\right)} e^{i\frac{1a}{2b}p^2} e^{-i\frac{p}{b}\omega} e^{i\frac{1d}{2b}\omega^2} dp$$

$$\times \int_{-\infty}^{\infty} g\left(t - u + \frac{q}{2}\right) \overline{g\left(t - u - \frac{q}{2}\right)} \times e^{i\frac{1a}{2b}q^2} e^{-i\frac{q}{b}\omega} dq e^{i\frac{a}{b}(4u(t-u))} du. \quad (24)$$

Applying the definition of the LCAF (6) and multiplying both sides of (24) by $e^{i\frac{1d}{2b}\omega^2}$ yields

$$\mathcal{AF}_{f\lozenge g}^{A}(t,\,\omega) = \int_{-\infty}^{\infty} \mathcal{AF}_{f}^{A}(u,\,\omega)\mathcal{AF}_{g}^{A}(t-u,\,\omega)e^{-i\frac{1d}{2b}\omega^2}e^{-i\frac{a}{b}(4u(t-u))}du.$$

3.3. Practical signal and simulation

In this subsection, we shall discuss an example how to compute the convolution of the LCAF in (21). We first use a Gaussian function

$$f(t) = (\pi\sigma^2)^{-1/4}e^{-t^2/2\sigma^2}.$$

Applying equation (20), we easily obtain

$$\mathcal{AF}_{f\lozenge f}^{A}(t,\,\omega) = \frac{e^{-i\frac{1d}{2b}\omega^2}}{2\pi bi}\frac{2\sigma^2}{2\sigma^2 a+b}e^{\omega^2\left(\frac{2c\sigma^2+id}{b-2ia\sigma^2}\right)}$$

$$\times(\pi\sigma^2)^{-1}\int_{-\infty}^{\infty} e^{-\frac{u^2}{\sigma^2}}e^{-(t-u)^2/\sigma^2}e^{-i\frac{a}{b}(4u(t-u))}du. \tag{25}$$

If $a = 0,$ then the above identity will reduce to

$$\mathcal{AF}_{f\lozenge f}^{A}(t,\,\omega)$$

$$= \frac{e^{-i\frac{1d}{2b}\omega^2}}{2\pi bi}\frac{2\sigma^2}{b}e^{\omega^2\left(\frac{2c\sigma^2+id}{b}\right)}(\pi\sigma^2)^{-1}\int_{-\infty}^{\infty} e^{-\frac{u^2}{\sigma^2}}e^{-(t-u)^2/\sigma^2}du$$

$$= \frac{e^{-i\frac{1d}{2b}\omega^2}}{2\pi bi}\frac{2\sigma^2}{b}e^{\omega^2\left(\frac{2c\sigma^2+id}{b}\right)}(\pi\sigma^2)^{-1}e^{-t^2/\sigma^2}\int_{-\infty}^{\infty} e^{-(2u^2-2tu)/\sigma^2}du.$$

Using the fact that

$$\int_{-\infty}^{\infty} e^{Ct^2\pm 2Dt}dt = \sqrt{\frac{\pi}{C}}e^{D^2/C},$$

where C and D are complex numbers satisfying $C \neq 0$ and $\operatorname{Re}(C) \geq 0.$ This gives

$$\mathcal{AF}^{A}_{f \lozenge f}(t, \omega) = \frac{e^{-i\frac{1d}{2b}\omega^2}}{2\pi bi} \frac{2\sigma^2}{b} e^{\omega^2\left(\frac{2c\sigma^2+id}{b}\right)} (\pi\sigma^2)^{-1} e^{-t^2/\sigma^2} \sigma \sqrt{\frac{\pi}{2}} e^{(2\sigma t)^2/2}.$$

If $a = 1,\ b = 1,$ then equation (25) becomes

$$\mathcal{AF}^{A}_{f \lozenge f}(t, \omega)$$

$$= \frac{e^{-i\frac{1}{2}d\omega^2}\sigma^2}{\pi i(2\sigma^2 + 1)} e^{\omega^2\left(\frac{2c\sigma^2+id}{1-2i\sigma^2}\right)} (\pi\sigma^2)^{-1} \int_{-\infty}^{\infty} e^{-2(u^2-tu)} e^{-i4(tu-u^2)} du$$

$$= \frac{e^{-i\frac{1}{2}d\omega^2}\sigma^2}{\pi i(2\sigma^2 + 1)} e^{\omega^2\left(\frac{2c\sigma^2+id}{1-2i\sigma^2}\right)} (\pi\sigma^2)^{-1} \int_{-\infty}^{\infty} e^{-(2-4i)u^2+2(t-i2t)u} du$$

$$= \frac{e^{-i\frac{1}{2}d\omega^2}\sigma^2}{\pi i(2\sigma^2 + 1)} e^{\omega^2\left(\frac{2c\sigma^2+id}{1-2i\sigma^2}\right)} (\pi\sigma^2)^{-1} \sqrt{\frac{\pi}{2-4i}}\, e^{(t-i2t)^2/2}.$$

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